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Analytical Solution of Transverse Electric and Transverse Magnetic Mode Electromagnetic Wave Equations in Left-Handed Materials

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Abstract:

Artificial composite materials that focus on electromagnetic waves are known as metamaterials. Metamaterials are artificial materials engineered by human technology, possessing a geometric structure built from microscopic, engineerable materials. The goal is for the new material to be able to direct light, sound, and waves, making it useful. Metamaterials are known as left-handed materials (LHMs), but the concept of metamaterials is broader than LHM. The purpose of this research is to solve the transverse electric (TE) and transverse magnetic (TM) electromagnetic wave equations in left-handed materials (LHMs) using the Nikiforov-Uvarov approach and to analyze the results of the energy spectrum equation from the solution of the transverse electric (TE) and transverse magnetic (TM) electromagnetic wave equations in left-handed medium (LHM). This research was conducted using Matlab software. The material being studied is the positive-negative gradient profile in an LHM medium thru variations in dielectric permittivity and/or magnetic permeability. Energy and wave equations were obtained with their visualization.

1. Introduction

Composite materials that focus on the study of electromagnetic waves are known as metamaterials. Metamaterials are human-engineered materials with a geometric structure. The goal is for the new material to be able to direct light, sound, and waves, making it useful. Metamaterials are known as left-handed materials (LHMs), but the concept of metamaterials is broader than LHMs, encompassing not only negative dielectric permittivity and magnetic permeability values but also the tendency to be designed for controlling incoming electromagnetic waves. Research on metamaterials was first conducted by Vaselago in 1968, who theoretically showed how, if a material's electromagnetic properties give it values for ε and μ [1]. In 2000, research was also conducted on materials with a negative refractive index that could be used to make perfect lenses [2].

The quantities of the basic characteristics of permittivity ε and permeability μ represent the dielectric response to electromagnetic waves. Proof of negative permittivity values was obtained in nanowires, while negative permeability was achieved with double split-ring resonators [2]. Experimentally reviewing LHM is indeed more difficult to achieve compared to theoretical reviews or experimental simulations thru modeling. Metamaterials are unique artificial composites with periodic or non-periodic structures, possessing distinct structural and material composition characteristics [3]. Metamaterial can provide negative refractive index values within a specific wavelength range, which is determined by the magnitude of the dielectric permittivity and magnetic permeability. Based on the values of dielectric permittivity and magnetic permeability, materials are classified into right-handed metamaterial (RHM) ($\varepsilon > 0$ & $\mu > 0$), electric plasma ($\varepsilon < 0$ & $\mu > 0$), left-handed material (LHM) ($\varepsilon < 0$ & $\mu < 0$) and magnetic plasma ($\varepsilon > 0$ & $\mu < 0$). LHM material can be used for various applications, such as lens control, filtering, anti-reflection coating, perfect lenses, and more [4]. By utilizing the use of LHM for technological advancement, it becomes an interesting area for researchers to develop, both theoretically and experimentally.

Metamaterials, which are artificial materials with electromagnetic properties not found in nature, offer the ability to control electromagnetic waves in unique ways. Research into the theoretical analysis, numerical, and mathematical study of electromagnetic waves in metamaterials is crucial in the context of modern technological development. Metamaterials, which are artificial materials with electromagnetic properties not found in nature, have the ability to control electromagnetic waves in unique and innovative ways. Previous research has shown that metamaterials can be used in a variety of applications, ranging from miniaturized antennas to more efficient sensors and communication devices [5]. Numerical and mathematical analysis plays a key role in understanding the behavior of electromagnetic waves in metamaterials. Methods such as Finite-Difference Time-Domain (FDTD) and the Finite Element Method (FEM) have been widely used to simulate the interaction between electromagnetic waves and metamaterial structures [6]. Additionally, research revealing the phenomenon of electromagnetic wave tunnelling occurring in metamaterials, demonstrating the ability of metamaterials to control and manipulate waves [7]. Furthermore, metamaterials also offer potential for applications in electromagnetic wave technology and control, such as how metamaterials can be designed to hide objects from electromagnetic wave detection, which is a significant breakthrough in the fields of military and civilian technology [8]. By leveraging the unique properties of metamaterials, researchers can design smaller and more efficient devices, which is highly relevant in the development of future communication technologies.

Overall, the theoretical study and numerical analysis of electromagnetic waves in metamaterial materials not only provide a better understanding of wave interaction in these media but also pave the way for innovation in various technological applications. Referring to previous research, such as the study conducted by Dalarson in 2005, which used the hyperbolic tangent function on permittivity and permeability, resulting in a transition from a right-handed to a left-handed system [9]. Then, in the same year, the study also analytically examined the propagation of right-handed and left-handed interface waves thru an index gradient [10] and material structures containing both positive and negative refractive indices tend to have a graded profile [11].

The Nikiforov-Uvarov method is an effective analytical technique for solving differential equations, including the Schrödinger equation, which is frequently used in quantum physics and electromagnetic applications. In the context of metamaterials, this method can be used to analyze the interaction of electromagnetic waves with complex metamaterial structures. However, the research by Fitriani and Suparmi focuses more on solving the Dirac equation for the spin-symmetric case and is not directly relevant to the analysis of electromagnetic waves in metamaterials [12].

2. Method

This research was conducted with the assistance of Matlab software. The material being studied is the positive-negative gradient profile in an LHM medium thru variations in dielectric permittivity and/or magnetic permeability. This research was conducted in several stages; schematically, the research stages are shown in the Figure 1.

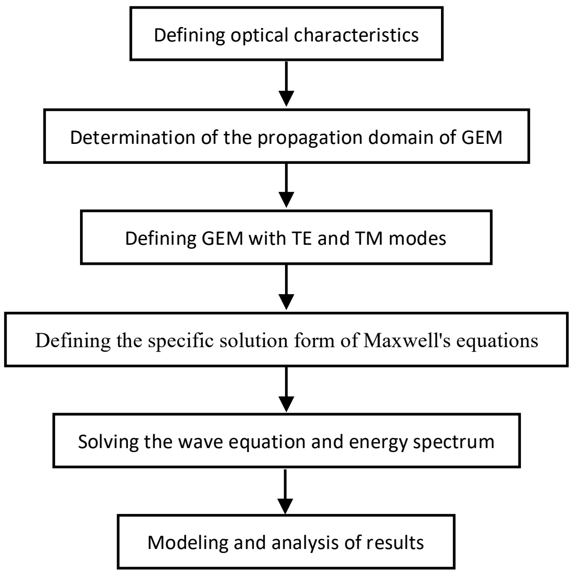

Figure 1. *Research Methods.*

The first step is to define the optical characteristics using the equations for dielectric permittivity and magnetic permeability, and then determine the propagation direction of the electromagnetic wave. After that, the TE and TM modes of the electromagnetic wave are determined using permittivity and permeability as hyperbolic functions.

3. Results and Discussion

This research is theoretical, using position and time functions with the Nikiforov-Uvarov method. The energy spectrum and wave function of electromagnetic waves are expressed in terms of dielectric permittivity and magnetic permeability as functions of position and time. Maxwell's equations in dielectrics require the value of free charge density to be zero. So the equation can be written as follows

$$\nabla \cdot \mathrm{D} = 0 \tag{1}$$

$$\nabla \times H = \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2}$$

By substituting the equations and solving analytically, we obtain

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\mu'}{\mu}\frac{\partial E_x}{\partial y} + \omega^2 \mu \varepsilon E_x = 0. \tag{3}$$

Using the variable separation technique will result in

$$E_{(x)} = E_{(y,z)} = E_{(y)} E_{(z)}. \tag{4}$$

Thus, we obtain

$$\frac{\partial^2 E_x}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(E_y E_z\right) \tag{5}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(E_y E_z\right) \tag{6}$$

$$\frac{\mu'}{\mu}\frac{\partial E_x}{\partial y} = \frac{\mu'}{\mu}\frac{\partial}{\partial y}\left(E_y E_z\right) \tag{7}$$

$$\frac{\mu'}{\mu}\frac{\partial E_x}{\partial y} = \frac{\mu'}{\mu}\frac{\partial}{\partial y}\left(E_y E_z\right). \tag{8}$$

To determine the wave function used, describing the motion of a free particle as a function of momentum and time, the electromagnetic wave equation can be transformed into a one-dimensional wave that is a function of linear momentum with the equation

$$\psi_{k(p_x,t)} = A \exp\left(i\frac{p_x}{\hbar}x - \frac{i}{\hbar}\frac{p_x^2}{2m}t\right) \tag{9}$$

$$\psi_{k(p_x,t)} = A \exp(ikx - iEt) \tag{10}$$

$$\psi_{k(p_x,t)} = A \exp\ i(kx - Et) = A \exp\ \pm i(kx - Et). \tag{11}$$

The equation is separated into 2 differential equations, yielding

$$\frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \tag{12}$$

and

$$\frac{\partial^2 E_{(y)}}{\partial y^2} - \frac{\mu'}{\mu}\frac{\partial E_{(y)}}{\partial y} + (\omega^2 \mu \varepsilon - k^2) E_{(y)} = 0 \tag{13}$$

where $\mu' = \frac{d\mu_{(y)}}{dy}$.

3.1. Permittivity and Permeability Gradation Based on Position Function

In the first study, the equations for the dielectric permittivity and magnetic permeability of the graded interface between RH and LH were defined as follows

$$\varepsilon_{(x)} = -\varepsilon_0[csch(\rho x) + \coth(\rho x)] \tag{14}$$

$$\varepsilon_{(x)} = -\varepsilon_0\left(\frac{1}{sinh(\rho x)} - \frac{cosh(\rho x)}{sinh(\rho x)}\right) \tag{15}$$

$$\varepsilon_{(x)} = -\varepsilon_0\left(\frac{1}{sinh(\rho x)} - \frac{cosh(\rho x)}{sinh(\rho x)}\right) \tag{16}$$

$$\mu'_{(x)} = -\mu_0\left(-\frac{\rho\ cosh(\rho x)}{sinh^2(\rho x)}\right) \tag{17}$$

$$\frac{\mu'_{(x)}}{\mu_{(x)}} = -\frac{\rho\ cosh(\rho x)}{sinh(\rho x)}. \tag{18}$$

From equation (13), we obtain

$$\frac{\partial^2 H_{(x)}}{\partial x^2} - \frac{\mu'}{\mu}\frac{\partial H_{(x)}}{\partial x} + (\omega^2 \mu \varepsilon - k^2) H_{(x)} = 0. \tag{19}$$

Substituting the equations for dielectric permittivity and magnetic permeability into equation (3), we get

$$\frac{\partial^2 H_{(x)}}{\partial x^2} + \frac{\rho \cosh(\rho x)}{sinh(\rho x)} \frac{\partial H_{(x)}}{\partial x} + \left\{ \omega^2 \left(\frac{-\mu_0}{sinh(\rho x)} \right) \varepsilon_0 \left(\frac{1}{sinh(\rho x)} + \frac{cosh(\rho x)}{sinh(\rho x)} \right) - k^2 \right\} H_{(x)} = 0. \tag{20}$$

With a very small value of ρx (namely $\rho x << 1$) approach $cosh\ (\rho x) \sim 1$ and $\sinh\ (\rho x) \sim \rho x$, we obtain

$$\frac{\partial^2 H_{(x)}}{\partial x^2} + \frac{1}{x} \frac{\partial H_{(x)}}{\partial x} + \left\{ \omega^2 \left(\frac{-\mu_0}{sinh(\rho x)} \right) \varepsilon_0 \left(\frac{1 + cosh(\rho x)}{sinh(\rho x)} \right) - k^2 \right\} H_{(x)} = 0 \tag{21}$$

$$\frac{\partial^2 H_{(x)}}{\partial x^2} + \frac{1}{x} \frac{\partial H_{(x)}}{\partial x} + \left(\frac{-\omega^2 \mu_0 \varepsilon_0 (1+1)}{(\rho x)^2} - k^2 \right) H_{(x)} = 0 \tag{22}$$

$$\frac{\partial^2 H_{(x)}}{\partial x^2} + \frac{1}{x} \frac{\partial H_{(x)}}{\partial x} + \left(\frac{\frac{-2\omega^2 \mu_0 \varepsilon_0}{\rho^2} - k^2 x^2}{x^2} \right) H_{(x)} = 0. \tag{23}$$

Then we solve it using a second-order differential equation, specifically using the Nikiforov-Uvarov method, the standard differential equation:

$$\frac{d^2 X(s)}{ds^2} + \frac{a_1 - a_2 s}{s(1 - a_3 s)} \frac{dX(s)}{ds} + \frac{(-\xi_1 s^2 + \xi_2 s - \xi_3)}{s^2 (1 - a_3 s)^2} X(s) = 0 \tag{24}$$

with ξ_1, ξ_2, ξ_3 is a system parameter that arises from variable separation.

3.2. Determining the energy spectrum

The value of the function in equation NU is determined to be

$$k \left(2n + 1 + \sqrt{\frac{8\mu_0 \varepsilon_0 \omega^2}{\rho^2}} \right) = 0. \tag{25}$$

From the energy equation obtained using the Nikiforov-Uvarov method, a graph was created with several variables:

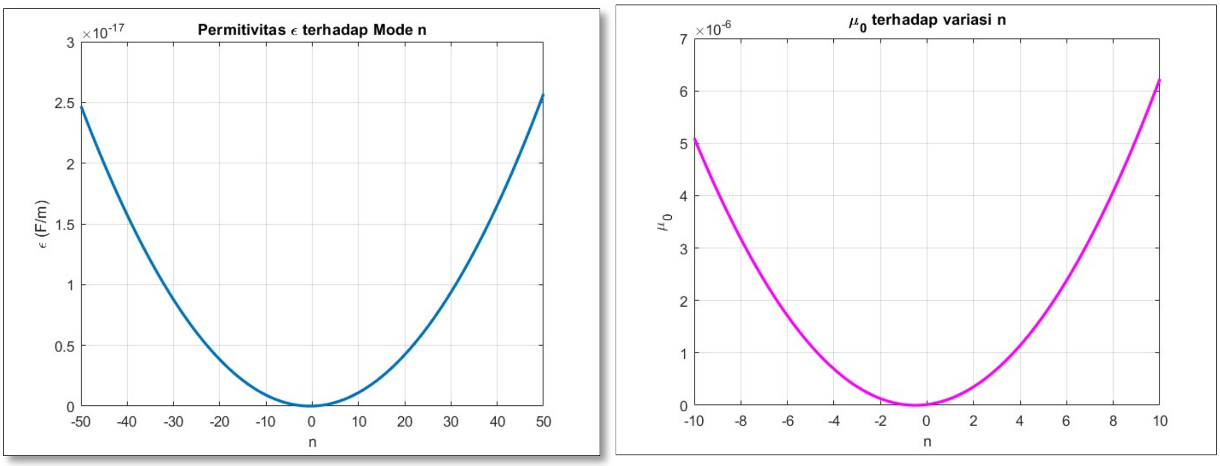

Figure 2. *graph of ε and μ vs n*

Figure 2 shows graph of ε_0 versus n, where ε_0 is the vacuum permittivity, n is the wave mode number, and μ_0 is the vacuum permeability. The graph shows a quadratic relationship between the permittivity value ε and the value of n, with the scale of ε being very small, on the order of $10 - 17\ F/m$, indicating that the material is a metamaterial. In metamaterial materials, the very small and even negative permittivity values indicate conditions where plasmons are in TE and TM modes. The graph of permeability versus n shows a quadratic relationship between the two. The minimum value occurs at $n = 0$ with a value of approximately $0.5 \times 10 - 6\, H/m$ and increases to $6.5 \times 10 - 6\, H/m$ at $n = \pm 10$. The results of this graph reinforce the theoretical model which shows that the values of ε and μ will increase quadratically by varying the value of n under the condition cut off ENZ (epsilon near zero) dan MNZ (miu near zero) [13].

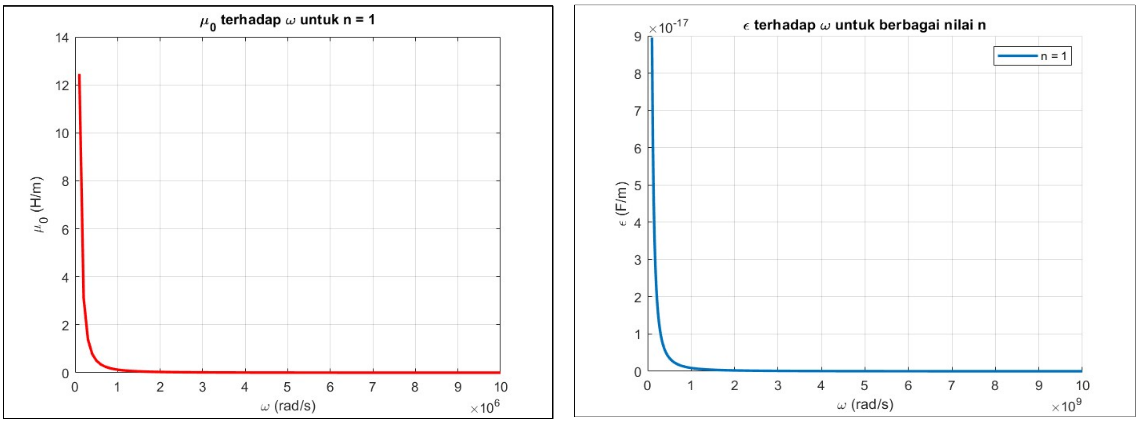

Figure 3. *graph of ε and μ vs ω*

Figure 3 shows the graphs of μ_0 versus ω and ε_0 versus ω, where μ_0 is magnetic permeability, ε_0 is vacuum permittivity, and ω is the frequency of the electromagnetic wave, show that the value of μ_0 is very high at low frequencies and decreases toward zero as ω increases. This shows that the magnetic response is very strong at the fundamental frequency but diminishes at high frequencies, which is a characteristic of metamaterials. The graph of ε_0 versus ω shows that the value of epsilon approaches zero or decreases as ω increases, which identifies a strong electrical response at low frequencies that weakens at high frequencies.

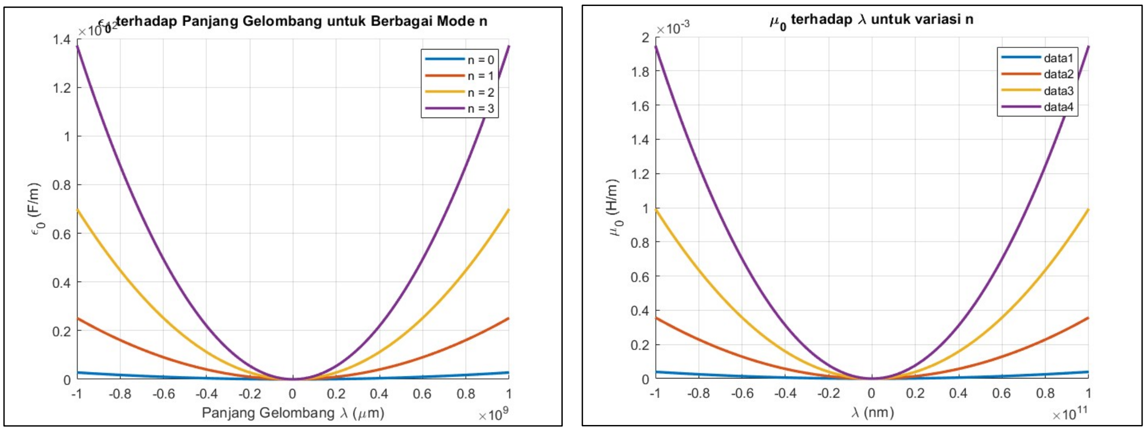

Figure 4. *graph of ε and μ vs λ*

Figure 4 shows the graph of ε_0 versus λ with varying values of n, where ε_0 is the vacuum permittivity and λ is the wavelength ranging from -1×109 to $1 \times 109\ \mu m$. All curves are parabolic in shape with a minimum value at $\lambda = 0$, then rise symmetrically for both $\lambda > 0$ and $\lambda < 0$. The value $n = 0$ has the smallest value, while $n = 3$ is the largest, which means that for higher n, a larger value of ε_0 is needed to produce waves at the same wavelength physically. For higher n values, more energy will be required, with higher permittivity. Wavelength determines the field distribution in a medium. The graph of μ_0 versus λ also has a minimum value as n increases, and the shape of the graph is also a symmetrical parabola with a minimum value at $n = 0$. For higher n, greater permeability is required for a specific wavelength, indicating that more magnetic energy storage is needed in a medium.

4. Conclusion

From the research conducted, the conclusions reached include that the permittivity (ε) and permeability (μ) in metamaterials can be negative or very small, with the values of ε and μ increasing quadratically with the wave mode number (n), indicating a greater energy requirement for higher modes. The electric and magnetic response of metamaterials is very strong at low frequencies but weakens at high frequencies, which is a typical characteristic of metamaterials. The electromagnetic wave equation was successfully solved analytically, and its energy spectrum depends on the values of ε_0, μ_0, and the grading parameter (ρ), all of which contribute to the control and manipulation of the wave.

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