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Hamilton Equations of Pendulum-Spring System

N W Rini* and **J Saefan**

Department of Physics Education, University of PGRI Semarang, Indonesia

*E-mail: rininurwidya@gmail.com

Abstract. The pendulum-spring system was studied by using Hamilton equations of motion. The total Hamiltonian of this system is complicated because of the complex mechanical system. The paper begins by introducing the physical system and defining a fixed coordinate system. Following this, the Lagrangian and Hamilton equations of motion are derived. Six equations of motion are obtained from the Hamilton equations in the form of ordinary differential equation. These equations can be employed for simulating the dynamical behavior of the system. The primary goal of this paper is to familiarize physics students with the Hamilton equations of motion as applied to the pendulum-spring system.

Keywords: Hamiltonian, pendulum, equation of motion, ODE

1. Introduction

Analytical mechanics, describing the motion of classical point particles, serves as the foundational model within theoretical physics. Starting from Newton's laws of dynamics, Hamilton formulated a variational principle that identifies classical trajectories as the stationary points of the action functional. Moreover, the action functional satisfies the Hamilton-Jacobi equation, a nonlinear partial differential equation [1]. The visualization of these concept is in Figure 1. It this paper, we study the Hamilton's equation of motion of spring-pendulum system. The Hamiltonian can be constructed through a Legendre transformation of the Lagrangian in classical mechanics [2]. Hamilton's equations consist of a set of coupled first-order differential equations, while the Lagrangian formalism yields a single set of second-order differential equations. The Hamiltonian formulation offers inherent advantages over the Lagrangian approach, as a coupled system of first-order differential equations is often numerically more stable and easier to solve compared to a single set of second-order differential equations [3]. For advanced studies, Mandal et al. demonstrated the prediction of the dynamical system by intergrating Hamiltonian and Lagrangian neural network. [4]

Udwadia discussed simple and systematics approach for getting equation of motion for the constrained double pendulum [5]. In addition, Biglari et al. discussed the dynamics of double pendulum system using Lagrangian and Hamiltonian formalism and numerical methods. They found out that the system is governed by a set of coupled non-linear ordinary differential equations [6]. Moreover Stachowiak et al. analysed the double pendulum numercially using a modified mid point integrator [7]. In another research, Indiati et al. studied the Hamilton equation on double pendulum with acial forcing constraint to obatin the equation of motion [8]. Furthermore, Griffin et al. studied the connection between a combinatorial aspect of the graph structure and chaotic behavior of the time-evolving strategy for two strategy game which possesses a generalized Hamiltonian on arbitrary graph sturcture [9]. While Azuaje discussed the solution of the Hamilton equations for time dependent hamiltonian system via solvable lie algebras of symmetries [10].
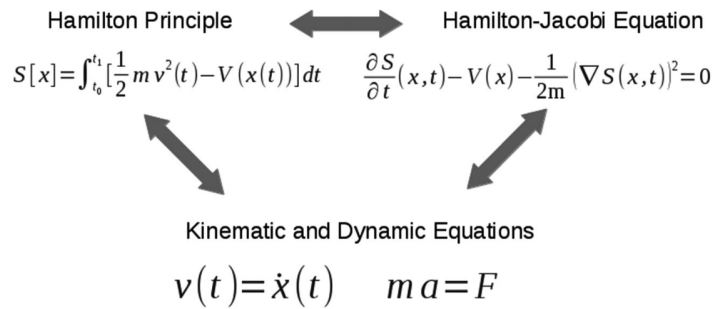
Hamilton Principle ⟷ Hamilton-Jacobi Equation

$$S[x] = \int_{t_0}^{t_1} \left[\frac{1}{2} m v^2(t) - V(x(t))\right] dt \qquad \frac{\partial S}{\partial t}(x,t) - V(x) - \frac{1}{2m}(\nabla S(x,t))^2 = 0$$

Kinematic and Dynamic Equations

$$v(t) = \dot{x}(t) \qquad m a = F$$

Figure 1. The triangle of analytical mechanics, linking Hamilton's principle, kinematic and dynamic laws and the Hamilton-Jacobi equation [1].

The aims of this paper were derived the Hamiltonian equations of motion and analyzed the dynamical behavior of the pendulum-spring system. The pendulum-spring system concisted of two masses illustraded in the Figure 2. The string l_1 connected to the mass m_1 was considered massless and inextensible. The second mass m_2 connected to the spring with the length l_2 and extend te length by x. The swing angles θ_1 and θ_2 were the swing angles each pendulum makes with respect to the vertical line. The concrete steps to get the equation of motion usin the Hamiltonian method was writing down the Lagrangian, then went through a series of the other steps, and finally arrive back at th eEuler-Lagrange or equivalent equations [11].

The Hamilton H of the system equals to the total mechanical energy, that is

$$H = T + U \tag{1}$$

The generalized momenta p_i corresponding to each generalized coordinate q_i is given

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2}$$

By using the standart prescriotion for a Legendre transformation, we define H of the system written I terms of the Lagrangian

$$H = \sum_i p_i \dot{q}_i - L \tag{3}$$

Calculating the partial derivative of the equation (3) with respect to the generalized coordinate q_i obtains [12]

$$\frac{\partial H}{\partial q_i} = -\dot{p}, \qquad \frac{\partial H}{\partial p_i} = \dot{q}. \tag{4}$$

The purpose of solving the spring the pendulum-spring problem using Hamiltonian is not to gain the equation of motion in efficiency, but it could make students familiar with a branch of physics that has a numerous indispensable applications to other branches. In other branches in physics such a statistical mechanics and quantum mechanics, the Hamiltonian formalism is extremely helpful for calculating anything useful. By deriving the already studied Lagrangian equation of motion, we show the derivation to get the Hamiltonian equation of motion.
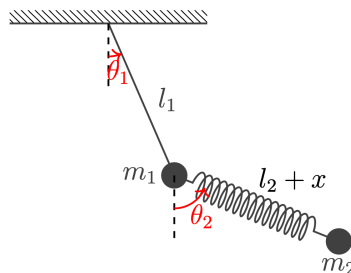
Figure 2. Pendulum-spring System

2. Research Methodology

This research worked with the theorictical methods where the Lagrangian $L(q, \dot{q}, t)$ was converted to the Hamiltonian $H(p, q, t)$ while preserving all the information. The first step is computing the kinetic energy T and potential energy V to formulate the Lagrangian L in terms of the coordinate q_i and their derivatives $\dot{q}_i$. Then, for each of the coordinates, we calculated the corresponding momenta $p_i = \partial L / \partial \dot{q}_i$. Subsequently, we calcute the Hamiltonian H as the sum of the products of p_i and $\dot{q}_i$ in terms of the q_i and p_i. This allowed us to formulate Hamilton's equation, where we obtained two equation for each generalized coordinates.

3. Results and Discussions

The Lagrangian for the pendulum-spring system, as described in [13], is given by

$$
\begin{aligned}
L = T - V \\
= \frac{1}{2} m_1 \left(l_1^2 \dot{\theta_1}^2 \right) + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1 + (l_2 + x)^2 \, \dot{\theta}_2 + \dot{x}^2 + 2 l_1 (l_2 + x) \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right. \\
- 2 l_1 \sin(\theta_1 - \theta_2) \, \dot{\theta}_1 \dot{x}) + (m_1 + m_2) g l_1 \cos \theta_1 + m_2 g (l_2 + x) \cos \theta_2 \\
- \frac{1}{2} k x^2
\end{aligned}
\tag{5}
$$

Where $\dot{\theta}_1 = d\theta_1/dt, \dot{\theta}_2 = d\theta_2/dt$, and $\dot{x} = x/dt$.

The generalized momenta related to the system are $p_{\theta_1}, p_{\theta_2}, p_x$. Decomposing these momenta according to the equation (5) yield

$$
p_{\theta_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 (l_2 + x) \cos(\theta_1 - \theta_2) \, \dot{\theta}_2 - m_2 l_1 \sin(\theta_1 - \theta_2) \, \dot{x} ,
\tag{6}
$$

$$
p_{\theta_2} = m_2 l_1 (l_2 + x) \cos(\theta_1 - \theta_2) \, \dot{\theta}_1 + m_2 (l_2 + x)^2 \dot{\theta}_2 ,
\tag{7}
$$

$$
p_x = -m_2 l_1 \sin(\theta_1 - \theta_2) \, \dot{\theta}_1 + m_2 \dot{x} ,
\tag{8}
$$

where $p_{\theta_1} = \partial L / \partial \dot{\theta}_1, p_{\theta_2} = \partial L / \partial \dot{\theta}_2$, and $p_x = \partial L / \partial x$.

The H is then given by the following expression

$$
\begin{aligned}
H = \frac{1}{2} m_1 \left(l_1^2 \dot{\theta_1}^2 \right) \\
+ \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + (l_2 + x)^2 \dot{\theta}_2^2 + \dot{x}^2 + 2 l_1 (l_2 + x) \cos(\theta_1 - \theta_2) \, \dot{\theta}_1 \dot{\theta}_2 \right. \\
- 2 l_1 \sin(\theta_1 - \theta_2)) + (m_1 + m_2) g l_1 \cos \theta_1 + m_2 g (l_2 + x) \cos \theta_2 - \frac{1}{2} k x^2
\end{aligned}
\tag{9}
$$

From the H of pendulum-spring system, a set of equation of motion was obtained which are equaivalent to the Euler-Lagrange equations

$$
\frac{\partial H}{\partial \theta_1} = -\dot{p}_{\theta_1}, \quad \frac{\partial H}{\partial \theta_2} = -\dot{p}_{\theta_2}, \quad \frac{\partial H}{\partial x} = -\dot{p}_x, \quad \frac{\partial H}{\partial p_{\theta_1}} = \dot{\theta}_1, \quad \frac{\partial H}{\partial p_{\theta_2}} = \dot{\theta}_2, \quad \frac{\partial H}{\partial p_x} = \dot{x}.
\tag{10}
$$

H as a function of the variables $\theta_1, \theta_2, x, p_{\theta_1}, p_{\theta_2}$ and p_x were required to solve the equation (10), so $\dot{\theta}_1, \dot{\theta}_2, \dot{x}$ and L were determined in terms of these variables. Gauss-Jordan Elimination method was used to get the first derivation of $\dot{\theta}_1, \dot{\theta}_2$, and x from equation (6) − (8), yield

$$
\dot{\theta}_1 = \frac{m_1 + m_2}{m_1^2 l_1^2} p_{\theta_1} - \frac{\cos(\theta_1 - \theta_2)}{m_1 l_1 (l_2 + x)} p_{\theta_2} + \frac{m_1 \sin(\theta_1 - \theta_2)}{m_1^2 l_1} p_x
\tag{11}
$$

$$
\dot{\theta}_2 = -\frac{\cos(\theta_1 - \theta_2)}{m_1 l_1 (l_2 + x)} p_{\theta_1} + \frac{m_1 + m_2 \cos^2(\theta_1 - \theta_2)}{m_1 m_2 (l_2 + x)^2} p_{\theta_2} - \frac{\sin 2(\theta_1 - \theta_2)}{2 m_1 (l_2 + x)} p_x
\tag{12}
$$

$$
\dot{x} = \frac{m_1 \sin(\theta_1 - \theta_2)}{m_1^2 l_1} p_{\theta_1} - \frac{\sin 2(\theta_1 - \theta_2)}{2 m_1 (l_2 + x)} p_{\theta_2} + \frac{m_1 + m_2 \sin^2(\theta_1 - \theta_2)}{2 m_1 m_2} p_x .
\tag{13}
$$

Then the equation (11), (12), and (13) substituted into equation (9) yields the H in terms of $\theta_1, \theta_2, x, p_{\theta_1}, p_{\theta_2}$ and p_x

$$H = \frac{1}{2m_1^2 l_1^2} p_{\theta_1}^2 + \frac{m_1 + m_2 \cos^2(\theta_1 - \theta_2)}{2m_1 m_2 (l_2 + x)^2} p_{\theta_2}^2 + \frac{m_1 + m_2 \sin^2(\theta_1 - \theta_2)}{2m_1 m_2} p_x^2 \tag{14}$$
$$- \frac{\cos(\theta_1 - \theta_2)}{m_1 l_1 (l_2 + x)} p_{\theta_1} p_{\theta_2} + \frac{m_1 \sin(\theta_1 - \theta_2)}{m_1^2 l_1} p_{\theta_1} p_x$$
$$- \frac{\sin(\theta_1 - \theta_2) \cos(\theta_1 - \theta_2)}{m_1 (l_2 + x)} p_{\theta_2} p_x - (m_1 + m_2) g l_1 \cos \theta_1$$
$$- m_2 g (l_2 + x) \cos \theta_2 + \frac{1}{2} k x^2$$

Equation (14) used on equation (10) to obtain the Hamiltonian equations of the pendulum-spring system, yield

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{x} \\ \dot{p}_{\theta_1} \\ \dot{p}_{\theta_2} \\ \dot{p}_x \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{l_1} \left(2\gamma_1 p_{\theta_1} - 2m_2 l_1 p_{\theta_2} + l_1 A p_x \right) \\ \frac{\alpha_1}{(l_2 + x)} \left(-2\gamma_1 p_{\theta_1} + 2l_1 \beta_1 p_{\theta_2} - 2B p_x \right) \\ \alpha_1 \left(A p_{\theta_1} - m_2 \alpha_2 p_{\theta_2} + 2\beta_2 \gamma_2 p_x \right) \\ \frac{\alpha_1}{(l_2 + x)} \left(\gamma_5 - m_2 C p_x^2 - 2m_2 \beta_3 \gamma_2^2 \sin \theta_1 \right) \\ \frac{\alpha_1}{(l_2 + x)} \left(-\gamma_5 + C p_x^2 - 2g\gamma_1 F \sin \theta_2 \right) \\ \frac{2\alpha_1}{(l_2 + x)^2} \left(l_1 \beta_1 p_{\theta_1}^2 - m_2 \gamma_3 p_{\theta_1} p_{\theta_2} - B p_{\theta_2} p_x + \gamma_4 \right) \end{pmatrix} \tag{15}$$

where $\alpha_n (n = 1, 2)$ defined according to

$$\alpha_1 = \frac{1}{2m_1 m_2 \gamma_2}, \qquad \alpha_2 = l_1 \sin 2(\theta_1 - \theta_2).$$

Meanwhile $\gamma_n (n = 1, 2, 3, 4, 5)$ were defined as

$$\gamma_1 = m_2 (l_2 + x), \qquad \gamma_2 = l_1 (l_2 + x), \qquad \gamma_3 = (l_2 + x) \cos(\theta_1 - \theta_2)$$
$$\gamma_4 = F(l_2 + x)(g \cos \theta_2 - m_2 k x), \qquad \gamma_5 = m_2 \alpha_2 p_{\theta_2}^2 - A p_{\theta_1} p_{\theta_2} - D p_{\theta_1} p_x - E p_{\theta_2} p_x$$

and $\beta_n (n = 1, 2, 3)$ were defined as

$$\beta_1 = m_1 + m_2 \cos^2(\theta_1 - \theta_2), \qquad \beta_2 = m_1 + m_2 \sin^2(\theta_1 - \theta_2), \qquad \beta_3 = m_1 (m_1 + m_2)$$

Last assumption made in the expression (14) are

$A = 2m_2 (l_2 + x) \sin(\theta_1 - \theta_2)$ $B = m_2 l_1 (l_2 + x) \sin(\theta_1 - \theta_2) \cos(\theta_1 - \theta_2)$
$C = 2l_1 (l_2 + x) \sin(\theta_1 - \theta_2) \gamma_3$ $D = 2\gamma_1 \gamma_3$
$E = 2m_2 \gamma_3 (1 - 2\cos^2(\theta_1 - \theta_2))$ $F = m_1 (l_2 + x) \gamma_2$

Equation (15) formed a set of coupled first order differential equations of motion on the variables $\theta_1, \theta_2, x, p_{\theta_1}, p_{\theta_2}$, and p_x. The behaviour of this system is visualized using Runge-Kutta fourth order method in Python.

The parameters set up for this system are $m_1 = m_2 = 1$ kg, $l_1 = l_2, g = 10$ m/s^2 and spring stiffnes $k = 10$ N/m. The initial values used are $\theta_1 = \pi/2°, \theta_2 = -\pi/2°, x = 0$ cm, $p_{\theta_1} = p_{\theta_2} = p_x = 0$ N/s. The simulation was made over the time interval $[0, 10]$ with $\Delta t = 0.0001$. The result of simulation was observed on Figure $3 - 8$.
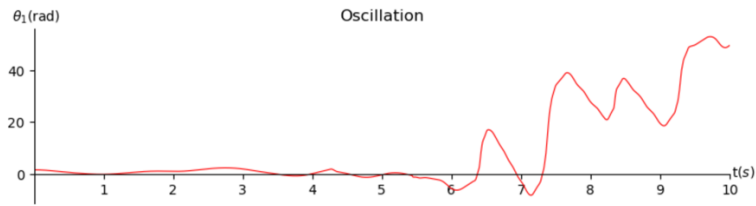
Figure 3. Oscillation θ_1 with respect to time t

Figure 2 shows that dynamical behavior of the first pendulum swung in the range $t = 0 - 6$ second and then 2π spinning clockwise indicated by the increasing position of θ_1. Subsequently the first pendulum 2π spinning back and forth indicated by the sine-like wave from the 8 to 9 second.
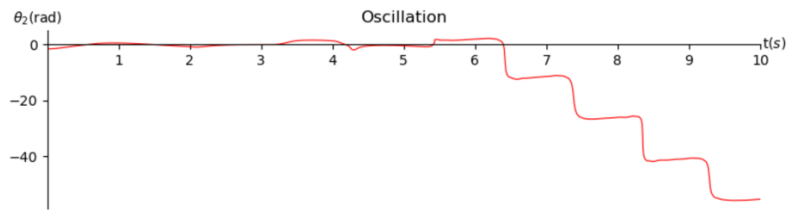

Figure 4. Oscillation θ_2 with respect to time t

The dynamical behavior of θ_2 in Figure 3 was like in Figure 2. The second pendulum swung for about 6 seconds and then swung counterclockwise from $t = 7 - 10$ seconds.

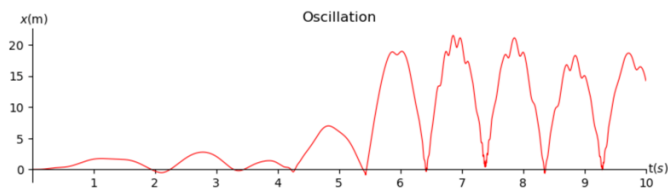**Figure 5.** Oscillation x with respect to time t

The position of x in Figure 4 leads to significantly more compicated dynamics due to the spring coupling between the two bobs of pendulum. The chaotic in Figure 4 shows similar behaviors with the dependence of the largest Lyapunov exponent for the spring Figure 3.5 studied by Ciaconel [14].

Figure 5 shows that the momentum p_{θ_1} changes over time. The peak of the graph indicated that the first pendulum moves with the highest velocity. Figure 5 shows stable solution for small amplitudes and chaotics behaviors appear in the range $t = 6 - 10$ seconds.
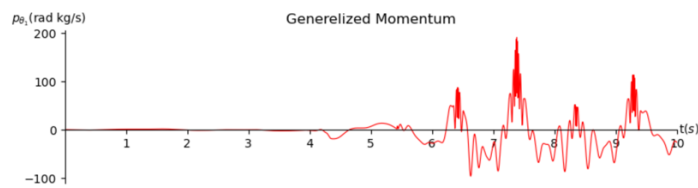

Figure 6. Momentum p_{θ_1} with respect to time t

A comparison with Figure 6 shows similar chaotic behavior in the range $t = 6 - 10$ seconds.
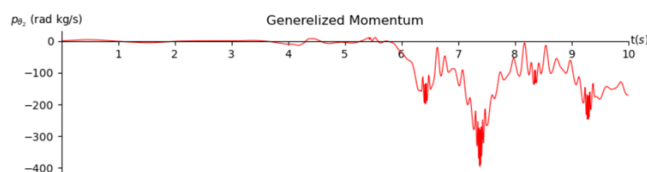

Figure 7. Momentum p_{θ_2} with respect to time t

However, Figure 7 has another range of chaotic behaviors in $t = 5 - 10$ seconds. This difference might arise due to the slack behavior of the pendulum-spring.
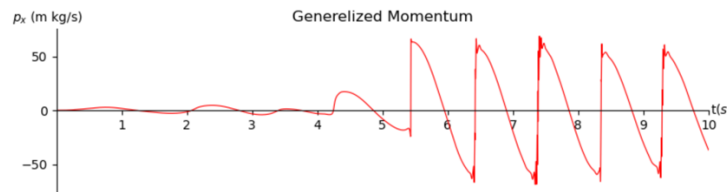

Figure 8. Momentum p_x with respect to time t

There are other ways to analyze the behavior of these mechanical system. Runge-Kutta is one of the numerical methods to see the complexity of the mechanical system by exposing the limit cycle, strange attractors, Poincare section, and bifurcation. Meanwhile, the focus of this paper is the derivation of the Hamiltonian of the pendulum-spring system. The Hamilton equation of motion obtained in this paper is the result of the manual derivation done by the authors. We are very grateful to readers and other researcher if there are any correction in the future research from the derivation and reduction we have made.

4. Conclusion

In this paper we have studied the Hamilton equations of motion in pendulum-spring system. Six equations of motion were derived on the given equation (15). The Hamiltonian described here is true in the sense that they allow us to rewrite the original equations through the standart Hamilton equations. The method described here is specifically suited for the systems which are described by the Lagrangians of the system.

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